Theory and Methodology

An algorithm of global optimization for solving layout problems

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Abstract

The two-dimensional layout problem is known to be NP-complete, and the current research work is basically in the heuristic way. In this paper, we mainly discuss the methods for solving layout problem about the artificial satellite module by virtue of graph theory and group theory. Also, an algorithm of global optimization is presented first time. The method given here can be extended to solve other type of layout problems. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The layout problem for apparatus modules of artificial satellite is to find an optimal strategy for installing on a base board (just cycloid domain) a finite number of apparatus (called graph elements or pieces) with possible different shapes, sizes and mass quantities. Usually, the shapes of pieces may be simply regarded to be two-dimensional. Graph elements with rectangular shape are considered in this paper. The scheme of installation is required to satisfy the following three constraints: (i) capacity constraints: pieces do not exceed the region of the domain, (ii) non-overlap constraints: pieces are not overlapped by each other, (iii) static non-equilibrium constraints: the distance between the centre of the base board and centroid of all the pieces should be minimized.

A great deal of work devoted to the layout problems has been done for many years because of the great significance both in theory and economy. The early researchers put their efforts mainly on the construction of algorithms, such as the heuristic algorithm, the man–computer mutual algorithm, the quasi-physical method in accordance with specific problems, etc [1]. In recent years, various mathematical models about this kind of problems have been established and their properties have been discussed [2]. There are also some researchers who use optimization theories to construct some algorithms for the layout optimal problem [3], but these algorithms are not satisfactory in some aspects, e.g., (i) the same objective value can often be obtained from different initial points, but whether it is the optimal solution
cannot be assured; (ii) these algorithms are lack of positive terminal criterions, and the final solution is often determined by the time of performance or by the number of different initial points but not by the problem itself.

A solution, usually is a local optimal solution, is obtained through a lot of computation, but the solution cannot be affirmed to be a global optimal solution because of the lackness of effective and convenient optimality conditions. The layout problem is NP-complete, and it is very difficult to construct descent direction and also requires a large amount of computation in determining non-overlapping graph elements. The emphases of this paper are put on the classifying of layout schemes. The layout problem about artificial satellite module is

2. Description of layout problem

Assume that \( n \) different rectangular pieces \( F_i \) are to be packed in a circle \( B \) (see Fig. 1). Where \( B = \{ x \in \mathbb{R}^2 : ||x|| \leq r_0 \} \), and \( r_0 \) (> 0) is the radius of \( B \). Denote the centre of the \( i \)th rectangular piece \( F_i \) by \( x_i \in \mathbb{R}^2 \), the direction of \( F_i \) by \( u_i \in \mathbb{R}^2 \), where \( u_i \) parallels the long edge of \( F_i \), and the angle between \( x_i \) and \( u_i \) is no more than 90°, where \( ||u_i|| = 1 \). Denote the half length of the two edges of \( F_i \) by \( a_{i1} \) and \( a_{i2} \), respectively, with \( a_{i1} \geq a_{i2} > 0 \). Define \( a_i = (a_{i1}, a_{i2}) \). \( v_i \) is a unit vector orthogonal to \( u_i \); \( u_i \) and \( v_i \) satisfy the right-handed law. Such a rectangular piece \( F_i \) can be uniquely determined by vectors \( x_i, u_i, a_i \in \mathbb{R}^2 \) as

\[
F_i = F(x_i, u_i, a_i) = \{ y = x_i + t_1u_i + t_2v_i : 1 \leq k \leq 2 \}.
\]

Assume that the centroid of piece \( F_i \) coincides with its centre of form, and its mass is \( m_i \in \mathbb{R}^+ \), \( i \in I_n = \{ 1, 2, \ldots, n \} \). For a specific layout problem, \( n, r_0, a_i, m_i \) are known to be constants, and thus the key point is to determine \( x_i, u_i \in \mathbb{R}^2, i \in I_n \).

**Definition 1.** Let

\[
y_i = (x_i, u_i) \in \mathbb{R}^4, \quad Y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^{4n},
\]

and \( F_i \) is defined by formula (1). Then \( Y \) is called a **layout scheme**.

Introduce a measure function of piece \( F_i \) as follows by virtue of the set-valued mapping:

\[
f_{\mu i} = f_{\mu i}(F_i) = f_{\mu i}(x_i, u_i) = \max \{ ||x|| : x \in F_i \}, \quad i \in I_n,
\]

\[
f(Y) = \max \{ f_{\mu 1}, f_{\mu 2}, \ldots, f_{\mu n} \}.
\]

The capacity constraint of the layout problem is defined by

\[
f(Y) \leq r_0 ~ \text{or} ~ \bigcup_{i=1}^{n} F_i \subset B.
\]

It has been proved that \( f(Y) \) is convex and non-differentiable (see [2]). So the capacity constraint (4) is convex one.

**Definition 2.** For a layout scheme \( Y \), if all pieces \( F_i \) do not mutually overlap, i.e.

\[
\text{int } F_i \cap \text{int } F_j = \emptyset, \quad i \neq j, \quad i, j \in I_n,
\]

where \( \text{int } F_i \) and \( \text{int } F_j \) are the internals of \( F_i \) and \( F_j \), respectively. Then the layout scheme \( Y \) is called a **non-overlap layout scheme**.

The static non-equilibrium quantity of the layout problem about artificial satellite module is
\[ H(Y) = \| \sum_{i=1}^{n} m_i x_i \| \quad \text{(see [4])}, \quad \text{and} \quad H(Y) \quad \text{is regarded as the objective function. So, the layout optimal problem (denoted by P) has the form} \]

\[
P: \quad \min \ H(Y) \\
\text{s.t.} \quad f_i(\int F_i \cap \int F_j = \emptyset, \ i \neq j, \ i, j \in I_n, \ Y \in \mathbb{R}^{4n}.}
\]

**Definition 3.** If all pieces \( F_i \) in layout scheme \( Y \) satisfy the capacity and non-overlap constraints ((4) and (5)), then \( Y \) is said to be a feasible layout scheme, or \( Y \) is called a feasible layout point of problem P.

Denote the set of all feasible layout schemes of problem P by \( D \), and the set of all non-overlap layout schemes by \( D_n \), i.e.,

\[ D = \{ Y \in \mathbb{R}^{4n} : Y \text{satisfies formulae (4) and (5)} \}, \]
\[ D_n = \{ Y \in \mathbb{R}^{4n} : Y \text{satisfies formula (5)} \}. \]

If \( D \neq \emptyset \), then the layout optimal problem P is said to be well posed.

**Definition 4.** Assume that \( Y \in D_n \). If pieces \( F_i \) and \( F_j \) satisfy one of the following conditions: (i) the edges of \( F_i \) and \( F_j \) coincide in some part; (ii) one of the vertices of \( F_i \) is on one of edges of \( F_j \); (iii) under the condition of non-overlap, translate \( F_i \) such that the centre of \( F_i \) on line segment between \( x_i \) and \( x_j \), during the translation some rotations are permitted, such that (i) or (ii) can be satisfied, then \( F_i \) and \( F_j \) are called adjoining pieces in layout scheme \( Y \), denoted by \( \langle F_i, F_j \rangle \).

**3. Layout scheme and its graph**

For any non-overlap layout scheme \( Y \in D_n \), define

\[ V = \{ F_1, F_2, \ldots, F_n \}, \]
\[ E(Y) = \{ (F_i, F_j) : F_i, F_j \in V, F_i \text{ and } F_j \text{ are adjoining pieces} \}, \]
\[ G(Y) = (V, E(Y)). \]

\( V \) is called the set of pieces of layout problem \( P \), \( E(Y) \) is the edge set of layout scheme \( Y \), and \( G(Y) \) is called the graph of layout scheme \( Y \). If \( Y \in D \), then \( G(Y) \) is called a feasible layout graph, and if \( Y \in D_n \), then \( G(Y) \) is called a non-overlap layout graph. Denote the set of all feasible layout graphs of problem P by \( G \), and all non-overlap layout graphs of problem P by \( G_n \), i.e.,

\[ G = \{ G(Y) : Y \in D \}, \]
\[ G_n = \{ G(Y) : Y \in D_n \}. \quad (6) \]

**Definition 5.** For the layout problem P, if graph \( G(Y_1) = (V, E(Y_1)) \) and graph \( G(Y_2) = (V, E(Y_2)) \) satisfy

\[ E(Y_1) = E(Y_2), \quad Y_1, Y_2 \in D_n, \quad (7) \]

then \( G(Y_1) \) and \( G(Y_2) \) are called equivalent graphs, denoted by \( G(Y_1) \equiv G(Y_2) \).

In graph \( G(Y) \) of layout scheme \( Y \in D_n \), the set of all adjoining pieces of \( F_i \) is denoted by \( V_{F_i}(Y) \), i.e.,

\[ V_{F_i}(Y) = \{ F_j : (F_i, F_j) \in E(Y), F_j \in V \}. \]

Denote the number of elements in \( V_{F_i}(Y) \) by \( |V_{F_i}(Y)| \). Actually, \( |V_{F_i}(Y)| \) is the degree (or rank) of \( F_i \) in layout scheme \( Y \in D_n \).

From the above definitions and corresponding relations between layout schemes and graphs, we have the following conclusion.

**Property 1.** Layout scheme \( Y \) is a non-overlap layout scheme if \( Y \) satisfies formula (5), \( G(Y) \) is a connected graph, and \( 1 \leq |V_{F_i}(Y)| \leq n - 1, i \in I_n. \)

**Definition 6.** Assume that \( Y_1, Y_2 \in D_n \). If the corresponding graph \( G(Y_1) \) and graph \( G(Y_2) \) are equivalent, then \( Y_1 \) and \( Y_2 \) are called isomorphic layout schemes, otherwise \( Y_1 \) and \( Y_2 \) are called non-isomorphic layout schemes. Let

\[ I(Y_1) = \{ Y : E(Y) = E(Y_1), \ Y \in D_n \}. \quad (8) \]

\( I(Y_1) \) is called the isomorphic layout equivalent class of layout scheme \( Y_1 \in D_n \), and any element in \( I(Y_1) \) is called its representative element.
Denote by \( I_e \) the set of all isomorphic layout equivalent classes of problem \( P \), i.e.,

\[
I_e = \{ I(Y) : Y \in D_n \}.
\]

Obviously, \( |I_e| \) is the number of all non-isomorphic layout schemes of problem \( P \).

**Property 2.** For any \( Y_1, Y_2 \in D_n \), if \( Y_1 \not\subset I(Y_2) \), then \( I(Y_1) \cap I(Y_2) = \emptyset \); otherwise if \( Y_1 \subset I(Y_2) \), then \( I(Y_1) = I(Y_2) \), \( E(Y_1) = E(Y_2) \) and \( G(Y_1) = G(Y_2) \).

**Property 3.** \( \bigcup_{Y \in D_n} I(Y) = D_n \), where \( Y \) runs over the representative elements in non-isomorphic layout equivalent classes.

**Property 4.** There is an one-to-one correspondence between the elements in \( G_n \) and in \( I_e \), so that \( |G_n| = |I_e| \).

### 4. Orbit and isomorphic layout equivalent classes

Define

\[
W = \{(F_i, F_j) : F_i, F_j \in V\}.
\]

It follows from \( |V| = n \) that

\[
|W| = \binom{n}{2} = \frac{n(n-1)}{2}.
\]

Let \( A = \{0, 1\} \). Define a set \( E_g \) and a graph \( G_g \) by mapping \( g : W \to A \):

\[
E_g = \{(F_i, F_j) : F_i, F_j \in V, \ g(F_i, F_j) = 1\},
\]

\[
G_g = (V, E_g).
\]

Let \( M = \{ g : W \to A \} = A^W \), \( X = \{ G_g : g \in M \} \). Obviously, \( X \) is the set of all graphs on the set of pieces \( V \), and the following relation holds:

\[
|X| = |M| = |A|^{|W|} = 2^{n(n-1)/2}.
\]

**Theorem 1.** Assume that layout problem \( P \) is well posed, then \( G_n \) is the proper subset of \( X \); the number of non-isomorphic layout scheme \( |I_e| \) is finite, and satisfies

\[
0 < |I_e| < 2^{n(n-1)/2}.
\]  \hspace{1cm} (9)

**Proof.** For any \( Y \in D_n \), define a mapping \( g : W \to A \) by

\[
g(F_i, F_j) = \begin{cases} 1 & \text{if } F_i \text{ and } F_j \text{ are adjoining pieces,} \\ 0 & \text{else,} \end{cases}
\]

where \( i \neq j \), \( i, j \in I_n \). Then, for any \( Y \in D_n \), there exists \( g \in M \) such that \( E(Y) = E_g, G(Y) = (V, E(Y)) = G_g = (V, E_g) \in X \), so that \( G_n \subset X \).

Since problem \( P \) is well posed, \( G_n \neq \emptyset \). Assume that \( g_0 \in M \) and satisfies

\[
g_0(F_i, F_j) = 0, \forall F_i, F_j \in V, i \neq j, \ i, j \in I_n
\]

Then the graph \( G_{g_0} = (V, \emptyset) \) defined by \( g_0 \) is in \( X \), i.e., \( G_{g_0} \in X \). It follows from Property 1 that \( G_{g_0} \notin G_n \). So \( G_n \) is a proper subset of \( X \), and we have

\[
0 < |G_n| < |X| = 2^{n(n-1)/2}.
\]

\[
|G_n| = |I_e|,
\]

so formula (9) is true. \( \square \)

Suppose that \( S_n \) is a symmetric group. For any \( \sigma \in S_n \), and any \( G_g = (V, E_g) \in X \), define the action of \( \sigma \) on graph \( G_g \) by

\[
\sigma(G_g) = (V, \sigma(E_g)),
\]

where \( \sigma(E_g) = \{ (\sigma(F_i), \sigma(F_j)) : (F_i, F_j) \in E_g \} \). Let the orbit of the symmetric group \( S_n \) acting on \( G_g \in X \) be \( X_{G_g} \), i.e.,

\[
X_{G_g} = \{ \sigma(G_g) : \sigma \in S_n \} \subset X.
\]  \hspace{1cm} (11)

\( \sigma \in S_n \) is a bijective mapping on \( V \), so all elements in orbit \( X_{G_g} \) are mutually isomorphic. By the Burnside lemma, the number of orbits in the set of graphs \( X \) determined by group \( S_n \) is

\[
N = \frac{1}{|S_n|} \sum_{\sigma \in S_n} |X(\sigma)|,
\]

where \( |S_n| = n! \), \( X(\sigma) \) is the set of stationary point of \( X \) at \( \sigma \in S_n \), i.e.,

\[
X(\sigma) = \{ G_g : \sigma(G_g) = G_g, G_g \in X \}.
\]  \hspace{1cm} (13)

It is known from properties of orbit that \( \{ X_{G_g} : G_g \in X \} \) is a partition of \( X \). The different orbits of \( S_n \) acting on \( X \) are denoted by \( X_1, X_2, \ldots, X_N \), and
\[ X = \bigcup_{k=1}^{N} X_k. \]  

(14)

**Property 5.** For layout problem \( P \), if there exists a graph \( G_{r_1} \in X_{G_{r}} \), such that \( G_{r_1} \) is a connected graph, then all elements in orbit \( X_{G_{r}} \), are connected graphs, the orbit \( X_{G_{r}} \) is called a connected orbit.

Denote the connected orbits in \( \{X_k\}_{k=1}^{N} \) by \( X_{ck} \), the number of \( X_{ck} \) by \( N_c \), the disconnected orbits by \( X_{dk} \), the number of \( X_{dk} \) by \( N_d \). Let

\[ X_c = \bigcup_{k=1}^{N_c} X_{ck}, \quad X_d = \bigcup_{k=1}^{N_d} X_{dk}. \]

Then

\[ X = X_c \cup X_d, \quad N = N_c + N_d. \]  

(15)

**Property 6.** Assume that layout problem \( P \) is well posed. If

\[ G_{g_1} = (V, E_{g_1}) \in X_{G_{r}}, k \in I_c = \{1, 2, \ldots, N_c\}, \]

then there exists \( Y_k \in D_n \), such that

\[ E(Y_k) = E_{g_1}, \quad G(Y_k) = (V, E(Y_k)) = G_{g_1} \subseteq G_n. \]

(16)

**Theorem 2.** Assume that \( Y \in D_n \), \( G(Y) = (V, E(Y)) \), and \( \pi = (F_1, F_j) \in S_n \) is a transposition (a cycle permutation of length 2) on \( V \). Then there exists \( Y_1 \in D_n \), such that

\[ E(Y_1) = \pi(E(Y)), \quad \pi(G(Y)) = G(Y_1). \]

(17)

**Proof.** Since \( Y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^{4n}, y_k = (x_k, u_k) \in \mathbb{R}^4, k \in I_n \), we can construct a layout scheme \( Y_1 = (y'_1, y'_2, \ldots, y'_n) \) from layout scheme \( Y \) as follows: (i) Let \( y'_i = y_i, \quad y'_j = y_i, \quad y'_k = y_k, \quad k \neq i, k \neq j, k \in I_n \). If \( Y_1 \in D_n \) and formula (17) holds, then the theorem is true, otherwise implement (ii). (ii) If we can translate or rotate \( F_1 \) or \( F_j \) such that \( Y_1 \) satisfies formulae (5) and (17) hold, then the theorem holds, otherwise implement (iii). (iii) Update the elements in \( V_{F_1}(Y_1) \) or \( V_{F_2}(Y_1) \), such that formulae (5) and (17) hold. By the definition of the adjoining pieces and there are no capacity constraint, \( Y_1 \) must satisfy formulae (5) and (17) after correcting pieces by the above translations or rotations. \qed

**Theorem 3.** Assume that \( G_{g_0} \in X_n, \) and \( X_{G_{r_0}} \) is the orbit of \( X \) acted by group \( S_n \). If there exists \( G_{g_0} = (V, E(Y_0)) \in X_{G_{r_0}}, \) such that \( Y_0 \in D_n \), then \( X_{G_{r_0}} \subseteq G_n. \)

**Proof.** For any \( G_{g_1} = (V, E(Y_1)) \in X_{G_{r_0}} \), it is known from the definition of orbit that there exists a permutation \( \sigma \in S_n \), such that

\[ \sigma(G_{g_0}) = G_{g_1}, \quad (V, E(Y_1)). \]

(18)

Permutation \( \sigma \) can be decomposed into the product of transpositions as

\[ \sigma = \pi_1 \pi_2 \cdots \pi_s. \]  

(19)

Let

\[ G_{g_1} = \pi_1 \pi_2 \cdots \pi_s(G_{g_0}) = \pi_1(G_{g_1}), \]

\[ k = s - 1, s - 2, \ldots, 3, 2. \]  

(20)

From formulae (18) and (19), we have

\[ \sigma(G_{g_0}) = \pi_1 \pi_2 \cdots \pi_s(G_{g_0}) = \pi_1(G_{g_1}) = G_{g_1}. \]

(22)

Since \( \pi_k \in S_n \), it follows from formulae (20) and (21) that \( G_{g_0} \in X_{G_{r_0}}, k = s, s - 1, \cdots, 2.1. \) According to Theorem 2 and formula (20), there exists \( Y_s \in D_n \), such that

\[ \pi_s(E(Y_0)) = E(Y_s), \quad G_{g_s} \subseteq G_n. \]

According to formulae (21) and (22), there exist \( Y_k \in D_n, k = s - 1, s - 2, \ldots, 3, 2, \) and \( Y_1 \in D_n \), such that

\[ \pi_{k}(E(Y_{k+1})) = E(Y_k), \quad G_{g_k} \subseteq X_{G_{r_0}}, \]

\[ k = s - 1, s - 2, \ldots, 3, 2, \]

\[ \sigma(E(Y_0)) = \pi_1(E(Y_2)) = E(Y_1), \quad G_{g_1} \subseteq G_n. \]

(23)

Since \( G_{g_1} \) is arbitrary taken in \( X_{G_{r_0}}, G_{g_1} \subseteq G_n. \) The proof is completed. \( \square \)

5. An algorithm for global optimization

By virtue of Property 5, Theorem 1 and formulae (14), (15), we get
From Property 6, we have \( X_c \subseteq G_n \), so \( G_n = X_c = \bigcup_{k=1}^{N_c} X_{ck} \). Assume that \( |X_{ck}| = N_k \). It follows from Property 4 that

\[
|I_c| = |G_n| = |X_c| = \sum_{k=1}^{N_c} N_k.
\]

Assume that orbit \( X_{ck} = \{G_{ck,i} : i = 1, 2, \ldots, N_k\} \).

From Property 6 we know that there exists a \( Y_{k,i} \in D_n \), such that \( G_{ck,i} = G(Y_{k,i}) \). Hence, the set \( \{Y_{k,i} : k = 1, 2, \ldots, N_c, i = 1, 2, \ldots, N_k\} \) is exactly the set of all non-isomorphic layout schemes of problem \( P \). By virtue of Property 3, we have

\[
D_n = \bigcup_{k=1}^{N_c} \bigcup_{j=1}^{N_k} \{I(Y_{k,i})\},
\]

i.e., finite non-isomorphic layout scheme equivalent classes will cover \( D_n \) and feasible domain \( D \subseteq D_n \) of problem \( P \).

In order to judge the feasibility of isomorphic layout equivalent class \( I(Y_{k,i}) \), compute

\[
f(Y_{k,i}) = \min \{f(Y) : Y \in I(Y_{k,i})\}.
\]

If \( f(Y_{k,i}) \leq r_0 \), then \( I(Y_{k,i}) \) is called an equivalent class of feasible isomorphic layout, i.e., \( I(Y_{k,i}) \cap D \neq \emptyset \). Solve problem \( P \) on every equivalent class of feasible isomorphic layout \( I(Y_{k,i}) \), and get their local optimal solutions. Since the number of such local optimal solutions is finite, we can obtain the global optimal solution of problem \( P \).

Based on the layout problem about artificial satellite module, an algorithm of global optimization can be constructed, denoted by PM. The main steps of PM are listed as follows:

**Algorithm PM**

1. **Step 1.** Input all known data of problem \( P \): \( r_0, n, m, a_i = (a_{i1}, a_{i2}), i \in I_n \). Set \( k = 1, A = 0 \) (\( A \) is a two-dimensional array with initial value zero). Compute the number of connected orbit \( |X_c| \), is denoted by \( N_c \).

2. **Step 2.** Compute \( N_k = |X_{ck}| \) (the number of elements in orbit \( X_{ck} \)), element \( G_{ck,i} \) and the corresponding layout scheme \( Y_{k,i} \in D_n \), such that

\[
X_{ck} = \{G_{ck,i} = G(Y_{k,i}) : i = 1, 2, \ldots, N_k\}
\]

Set \( i = 1 \).

3. **Step 3.** Compute

\[
f(Y_{k,i}) = \min \{f(Y) : Y \in I(Y_{k,i})\}
\]

If \( f(Y_{k,i}) \leq r_0 \), then \( Y_{k,i} \in D \), and go to step 6.

4. **Step 4.** Set \( i = i + 1 \). If \( i \leq N_k \), then go to step 3.

5. **Step 5.** Set \( k = k + 1 \). If \( k \leq N_c \), then go to step 2, otherwise go to step 7.

6. **Step 6.** Solve the following problem \( P_{ki} \) on \( I(Y_{k,i}^\prime) \), where \( Y_{k,i}^\prime \in D \) is an initial point.

7. **Step 7.** Solve problem \( P \) to obtain the global optimal solution and the optimal value \( H(Y_{k0,0}) \):

\[
H(Y_{k0,0}) = \min \{H(Y_{k,i}) : (k, i) \in A\}.
\]

**Example 1.** Suppose \( n = 3 \). Then \( V = \{F_1, F_2, F_3\} = \{1, 2, 3\}, |X| = 2^3 = 8 \). The edges \( E_{gi} \) defined by mapping \( g_i : W \rightarrow A, i = 1, 2, \ldots, 8 \), are:

\[
E_{g1} = \emptyset, E_{g2} = \{(1, 2), (2, 3), (1, 2, 3), (1, 2, 3, 1)\}, \quad E_{g3} = \{(3, 1, 2), (3, 1)\},
\]

respectively. Then \( X = \{G_{g1}, G_{g2}, G_{g3}\} \), \( X_{g1} = \{G_{g1}, G_{g2}, G_{g3}, G_{g4}\} \), and \( X_{g2} = \{G_{g1}, G_{g2}, G_{g3}\} \) and \( X_{g3} = \{G_{g1}\} \), respectively, where the connected orbits are \( X_c = X_{g1} \cup X_{g2} \) with \( N_1 = 3 \) and \( N_2 = 1 \). The numbers of all kinds of orbits are \( N = 4, N_c = 2 \) and \( N_k = 2 \). It follows from Property 6 that there exist \( Y_i \in D_n \), such that \( G_{gi} = (V, E(Y_i)) \), \( i = 5, 6, 7, 8 \).
\[ G_n = \{ G(Y_i) : i = 5, \ldots, 8 \}, \]
\[ I_c = \{ I(Y_i) : i = 5, \ldots, 8 \}, \]
\[ D_n = \bigcup_{i=5}^{8} I(Y_i). \]

Since \( D \subset D_n \), we can obtain its global optimal solution by using PM algorithm.

**Example 2.** Assume that
\[ n = 4, \ V = \{ F_1, F_2, F_3, F_4 \}, \ |X| = 64 \]

Firstly, determine \( E_{g_i}, i = 1, 2, \ldots, 64 \), which are corresponding to mapping \( g_i : W \rightarrow A \) by the similar method used in Example 1; determine all the elements in \( X \). Then compute the number of orbits on \( X \) determined by the symmetric group \( S_4 \). One obtains \( N = 11 \), orbits \( X_{g_k}, k = 1, 2, \ldots, N \), the number of connected orbits \( N_c = 6 \), the number of every connected orbit \( N_1, N_2, \ldots, N_6 \) and the elements \( G_{c,k,i} \) in connected orbits, \( k = 1, 2, \ldots, 6 \), \( i = 1, 2, \ldots, N_k \). Compute the layout scheme \( Y_{\hat{k},i} \in D_n \) corresponding to graph \( G_{c,k,i} \). Finally, we can obtain the global optimal solution of problem \( P \) by using PM algorithm.

6. For further reading

[5,6]

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References